Infinitesimals in Modern Mathematics

Jonathan W. Hoyle, III

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Introduction

- Who Am I?
- Eastman Kodak my employer (but I am here on my own accord)
- Overview of *Infinitesimals in Modern Mathematics*
- Paper & slides available at:

  http://www.jonhoyle.com/MAAseaway

  - Contact me at:

    jonhoyle@mac.com

Outline

- Terms & Definitions
- Hyperreals (Nonstandard Analysis)
  - Construction, Terminology
  - Results, Proofs
- Surreal Numbers
- Superreal Numbers [Tall]
- Super-real Numbers [Dales & Woodin]
  - Smooth Infinitesimal Analysis
  - Q & A

Terms & Definitions

- What is an infinitesimal?
- A non-zero number ε such that $|\varepsilon| < {}^{1}/_{n}$ for all $n \in \mathbf{N}$.
- No infinitesimals exist in $\mathbf{R}$
- Other terms in the paper:
  - Internal set
  - External set
  - Transfer Principle

Construction of *R

- Begin with $\mathbf{R}^{\infty}$, the set of ordered sequences of $\mathbf{R}$. Examples:

  $< 1, 0, 1, 0, 1, \dots >$

  $< 2, 3, 5, 7, 11, \dots >$

  $< -1, \pi, 0.0001, 10^{10}, \sqrt{17}, \dots >$

- Identify reals as a subset, eg: $3 = < 3, 3, 3, \dots >$
- Define arithmetic and extended functions:

$$a + b = < a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots >$$

$$a \times b = < a_0 \times y_0, a_1 \times b_1, a_2 \times b_2, \dots >$$

$$a \div b = < a_0 \div b_0, a_1 \div b_1, a_2 \div b_2, \dots >$$

$$a^b = < a_0^{b_0}, a_1^{b_1}, a_2^{b_2}, a_3^{b_3}, \dots >$$

$$f(a) = < f(a_0), f(a_1), f(a_2), \dots >$$

Equivalence Relation

- Divide subsets of $\mathbf{N}$ into “large” & “small”:
  - All finite subsets of $\mathbf{N}$ are “small”
  - All cofinite subsets of $\mathbf{N}$ are “large”
  - Complement of a “small” set is “large”, vice versa
- $< a_0, a_1, \ldots > = < b_0, b_1, \ldots >$ holds when the agreement set $< a_0 = b_0, a_1 = b_1, \ldots >$ is “large”.
- Using a *non-principal ultrafilter* on $\mathbf{N}$, we can define an equivalence relation
  - $^*\mathbf{R}$ is the set of equivalence classes over $\mathbf{R}^\infty$
  - $^*\mathbf{R}$ is a totally ordered field

Infinities Both Great & Small

- Ordered: x < y when the set $\{ i \mid x_i < y_i \}$ holds true for a "large" set of indices
- Identify r = $< r, r, r, \ldots > \textit{ for all } r \in \mathbf{R}$
- Let ω = $<1, 2, 3, \ldots>$
- We see that $\omega > n$, *for all* $n \in \mathbf{N}$
- Thus ω is an infinite element of $^*\mathbf{N}$
- Moreover, let ε = 1/ω = $<1, {}^1/_2, {}^1/_3, \ldots>$
  - Likewise $\varepsilon < r$, *for all* $r \in \mathbf{R}^+$
  - Thus ε is an infinitesimal.

Hyperreal Results

- For $x, y \in {}^*\mathbf{R}$, if $x - y$ is infinitesimal, we say that x *is infinitely close to* y, written $x \approx y$.
- The set of all hyperreals infinitely close to x is called *the halo of* x, written $hal(x)$.
- Every finite hyperreal x is infinitely close to ***exactly one*** standard real r.
- All finite hyperreals can be expressed as $r + i$ where $r \in \mathbf{R}$ and i is an infinitesimal.
    - ${}^*\mathbf{R}$ is not Dedekind complete.
    - i.e. ${}^*\mathbf{R}$ has "holes" in it.

Nonstandard Proofs

- Nonstandard proofs tend to be smaller and more intuitive than standard ones.
- Standard definition of continuity:

  A function f is *continuous* at x_0 if

  $$\forall \varepsilon > 0\ \exists \delta > 0 \ni: |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

- Nonstandard definition of continuity:

  A function f is *continuous* at x_0 if $x \approx x_0 \Rightarrow f(x) \approx f(x_0)$.

  - NSA definition much clearer
  - Paper compares two proofs

Surreal Numbers

- Reminicent of Dedekind cuts:
- Recursively defined as two sets of surreal numbers $\{ L \mid R \}$ such that $L > R$.
- $0 = \{ \mid \}$, $1 = \{ 0 \mid \}$, $-1 = \{ \mid 0 \}$
- $2 = \{ 0, 1 \mid \}$, $-2 = \{ \mid 0, -1 \}$, $1/2 = \{ 0 \mid 1 \}$
- $\omega = \{ 1, 2 \ldots \mid \}$, $\varepsilon = \{ 0 \mid \ldots 1/4, 1/2, 1 \}$
- $\sqrt{\omega} = \{ 1, 2, 3 \ldots \mid \ldots \omega/4, \omega/2, \omega \}$
  - Surreals very large, superset of even **On**, the class of ordinals

Superreal Numbers [Tall]

- Adjoins to $\mathbf{R}$ a single infinitesimal ε
- Generate others by closure over $+, -, \times, \div$
- The superreals $\mathfrak{R}$ is the set containing:

$$a_n\varepsilon^n + \ldots + a_1\varepsilon + a_0 + a_{-1}\varepsilon^{-1} + \ldots + a_{-m}\varepsilon^{-m},\ \forall a_k \in \mathbf{R}$$

- All infinitesimals of the form $\delta = a_1\varepsilon + \ldots + a_n\varepsilon^n$
- Analytic functions extended via power series:

$$sin\ \delta = \delta - \delta^3/3! + \delta^5/5! + \ldots$$

- Algebraic in nature (no First Order Logic required)
  - Leibniz-like Infinitesimal Calculus
  - $\mathfrak{R}$ is not as strong as $^*\mathbf{R}$

Super-real Numbers [Dales & Woodin]

- A more generalized abstract extension of $\mathbf{R}$:
  - Let $C(R)$ be the algebra of continuous functions
  - Let P be a *prime ideal* on $C(R)$
  - Let A be the *factor algebra* of $C(R)/ P$
  - Let F be a *quotient field* of A strictly containing R
- If F is not order-isomorphic to $\mathbf{R}$, then F is a *super-real* field.
  - If *prime ideal* P is the *maximal ideal*, then F is the field of the *hyperreal* numbers

Smooth Infinitesimal Analysis

- Infinitesimals are nilpotent: $\varepsilon^2 = 0, \forall \varepsilon \in \mathbf{I}$
- ε not invertible, so SIA does not form a field
- No infinite values in SIA
- Infinitesimal Cancellation Law:

$$\textit{If } x \cdot \varepsilon = y \cdot \varepsilon \textit{ then } x = y \quad \forall x,y \in \mathbf{R}$$

- All curves linear at infinitesimal granularity:

$$f(x + \varepsilon) = f(x) + f'(x) \cdot \varepsilon$$

- All functions continuous & differentiable!
  - SIA inconsistent with the *Law of the Excluded Middle.*

For more information…

Copies of the paper & slides:

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