An Introduction to Surreal Numbers

Jonathan W. Hoyle, III
Mathematical Association of America
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Outline

- Properties of Surreal Numbers
- Surreal Construction:
  - Conway Cuts
  - Sign Expansion
  - Axiomatic Treatment
- Surreal Arithmetic
- Comparison with Hyper-reals
- Calculus & Analysis on the Surreals
- Paper & slides available to view and download at

  http://www.jonhoyle.com/MAASeaway

  - Email: jonhoyle@mac.com

Properties of Surreal Numbers

- Discovered by Conway, popularized by Knuth, others.
- The surreals, denoted **No**, is an extension of **R**
- Totally ordered field (set-theoretically the *"largest"*)
- **No** is a proper class (too *"big"* to be a set)
- Dedekind Incomplete (has *"holes"* in it)
- Contains the following as subsets:
  - **R**, the set of reals
  - **On**, the class of ordinals
  - ***R**, the set of hyper-reals
  - $\mathscr{R}$, the set of super-reals
- Can make sense of unusual operations, like:

$$\log(\omega - 1) \qquad\qquad {}^{1}\!/\!_{\omega} + \sqrt[3]{(\pi - \aleph_{17})}$$

Surreal Construction: Conway Cuts

- Recursively defined as two sets (possibly empty) of surreal numbers $\{ L \mid R \} \ni: l < r \ \forall l \in L, r \in R$.
- Each iteration is the *birthday* of new surreal numbers
- Let S_n be the set of surreals created on birthday n:

S_0: $\{ \varnothing \mid \varnothing \} = \{ \mid \} = 0$

S_1: $\{ \mid 0 \} = -1$, $\{ 0 \mid \} = 1$, ~~$\{ 0 \mid 0 \}$~~

S_2: $\{ \mid -1 \} = -2$, $\{ -1 \mid 0 \} = -\tfrac{1}{2}$, $\{ 0 \mid 1 \} = \tfrac{1}{2}$, $\{ 1 \mid \} = 2$

$S_3 = \{ \pm\tfrac{1}{4}, \pm\tfrac{3}{4}, \pm 3 \}$, $S_4 = \{ \pm\tfrac{1}{8}, \pm\tfrac{3}{8}, \pm\tfrac{5}{8}, \pm\tfrac{7}{8}, \pm 4 \}$

- For $S_* = \bigcup S_n$, integers & dyadic rationals constructed
  - At S_ω, all remaining rationals & reals constructed

Surreal Construction: Conway Cuts

- Infinite & infinitesimal values:

$$\omega = \{\ 0, 1, 2, \ldots \mid\ \}$$

$$\varepsilon = \{\ 0 \mid \ldots \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, 1\ \}$$

$$\omega+1 = \{\ 0, 1, 2, \ldots, \omega \mid\ \}$$

$$\omega^2 = \{\ \omega, \omega\cdot 2, \omega\cdot 3, \ldots \mid\ \}$$

- Some unusual values:

$$\omega-1 = \{\ 0, 1, 2, \ldots \mid \omega\ \}$$

$$\omega/2 = \{\ 0, 1, 2, \ldots \mid \ldots \omega-2, \omega-1, \omega\ \}$$

$$\sqrt{\omega} = \{\ 0, 1, 2, \ldots \mid \ldots \omega/8, \omega/4, \omega/2, \omega\ \}$$

$$\log \omega = \{\ 0, 1, 2, \ldots \mid \ldots \sqrt[4]{\omega}, \sqrt[3]{\omega}, \sqrt{\omega}, \omega\ \}$$

Surreal Construction: Sign Expansion

- A surreal number is a function mapping an initial segment of ordinals to $\{ +, - \}$.
- Can be displayed as a string: **+-+**
- Let $x, y \in \mathbf{No}$. Let α be the smallest ordinal at which the strings for x, y differ. Then $x < y$ iff $x(\alpha) < y(\alpha)$, where - $<$ undefined $< +$. For example:

+-++- $<$ **+-++** $<$ **+-+++**

- For $x \in \text{No}$, $b(x)$ = length of the string.
- The ordinal ξ is simply **+++…+** (ξ pluses).
  - All integers and dyadic rationals have finite length
    - Other reals have length ω

Surreal Construction: Axiomatic

- The triplet $< \mathbf{No}, <, b >$ is a surreal number system iff:
  1. $<$ is a total order over $\mathbf{No}$
  2. b: $\mathbf{On} \rightarrow \mathbf{No}$ is onto (called the *birthday* function)
  3. $\forall$ A, B $\subseteq \mathbf{No} \ni : \forall x \in A, y \in B, x < y$,
     then $\exists!\, z \in \mathbf{No} \ni : b(z)$ is minimal and $x < z < y$.
  4. For $\alpha \in \mathbf{On}$, if $\alpha > b(x)\ \forall x \in A,B$, then $b(z) < \alpha$.

Surreal Arithmetic

- Standard operations defined: $+ \; - \; \times \; \div \; \sqrt{} \; exp \; log$
- Surreal operations behave like real operations:
  - Addition, multiplication commutative
  - Subtraction, division (except by 0) defined
  - different from ordinal & cardinal arithmetic
- Infinitesimals are inverses of transfinite "ordinals"
- Most elementary infinitesimal: $\varepsilon = 1/\omega$ (birthday ω)
- Sur-complex numbers can be defined as:

$$\{ a + bi \mid a, b \in \mathbf{No}, \; i = \sqrt{-1} \}$$

Comparison with *R

- **No** is far richer and more dense than ***R**
- Examples:
  - **No** contains all the ordinals
  - ***N** contains no ordinals larger than |***N**|
  - $|{*}\mathbf{R}| = \mathfrak{c}$
  - **No** is a proper class, and thus $|\mathbf{No}| > \aleph_\alpha \ \forall \alpha \in \mathbf{Ord}$
- Although some characteristic sequences of ***N** look similar, no element of ***N** is a countable ordinal:

$$\omega \neq < 0, 1, 2, \ldots >$$

Calculus & Analysis under **No**

- Let $\{ a_i \}_{i \in \mathbf{N}}$ be a sequence in $\mathbf{R}$. Then for any positive infinitesimal $x \in \mathbf{No}$, $\sum_{i \in \mathbf{N}} a_i x^i$ converges in **No**.
- **No** is *Dedekind Incomplete* and thus contains "holes", creating difficulties in defining integration in general.
- Specific cases can be defined, such as for polynomials:

  Let $p(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial with $a_i \in \mathbf{No}$.

  Then: $\int_0^x p(t)dt = \sum_{i=0}^{n} a_i x^{i+1}/(i+1)$
- Similarly, analytic functions can be integrated.
- However, others fail, such as *exp()*.
  - Surreal integration fails translation-invariance.

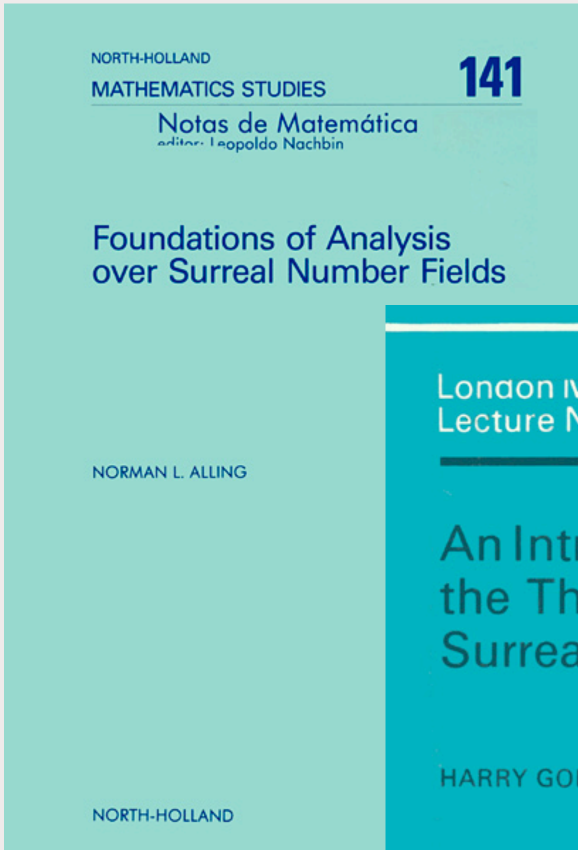
Calculus & Analysis under ξ**No**

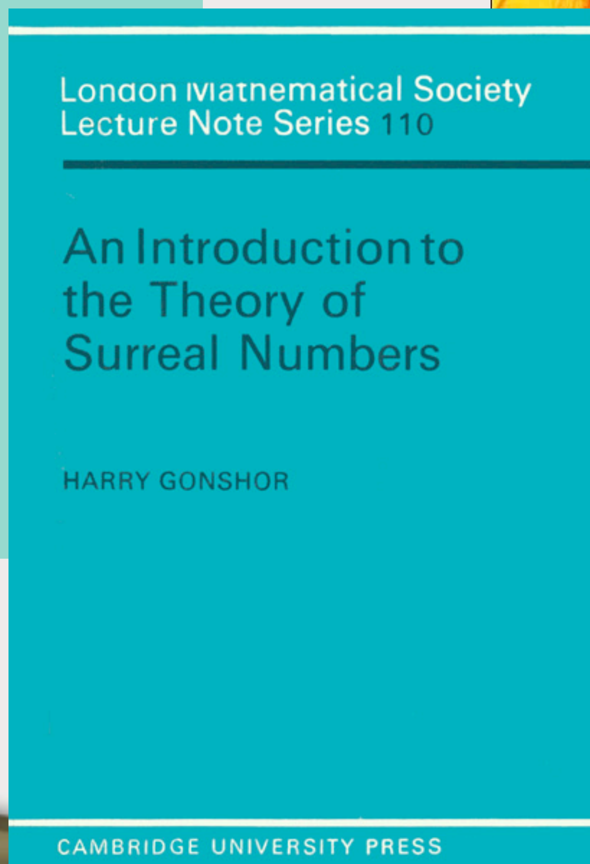
- Choose some large $\xi \in$ **On** for which $\aleph_\xi$ is regular.
- Let ξ**No** be a subfield of **No** ∋: $\forall x \in$ ξ**No**, $b(x) <= \xi$.
- ξ**No** is a therefore real closed field which is a complete binary tree of height $\aleph_\xi$. (Note ξ**No** is a set.)
- Furthermore, ξ**No** is an η_ξ-set, and can be described easily in terms of its natural power series structure.
- This is the approach taken by Norman Alling in *Foundations of Analysis over Surreal Number Fields*

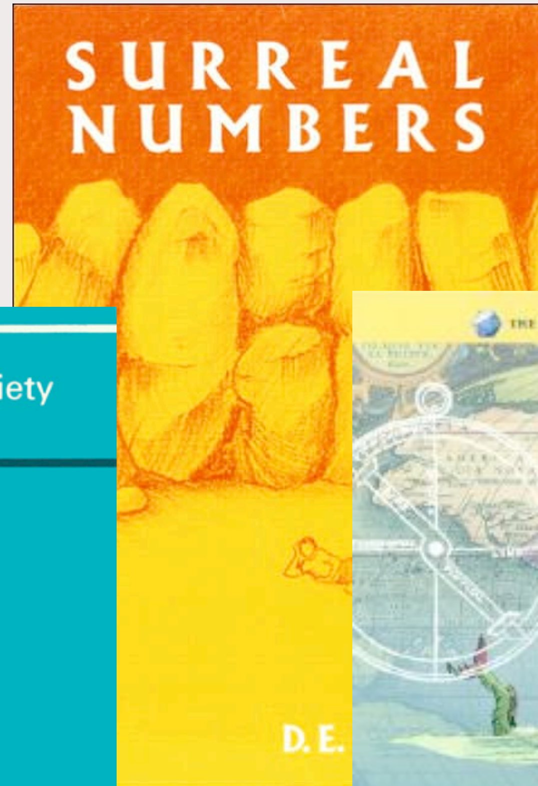
Conclusion

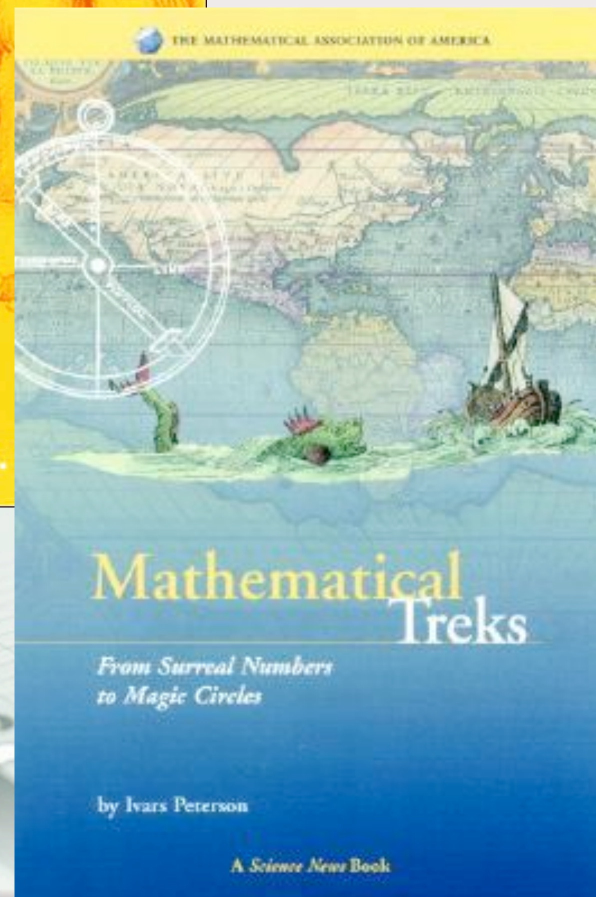
- **No** is an extension of **R** containing both infinite and infinitesimal values
- **No** is a necessarily *incomplete* ordered field
- This field combines disparate sets (hyper-reals, ordinals, etc.) into a single cohesive proper class.
- It is the "largest" theoretically possible ordered field.
- Many exciting research opportunities exist for those wishing to further the advance of **No**.

Further Reading…









For more information…

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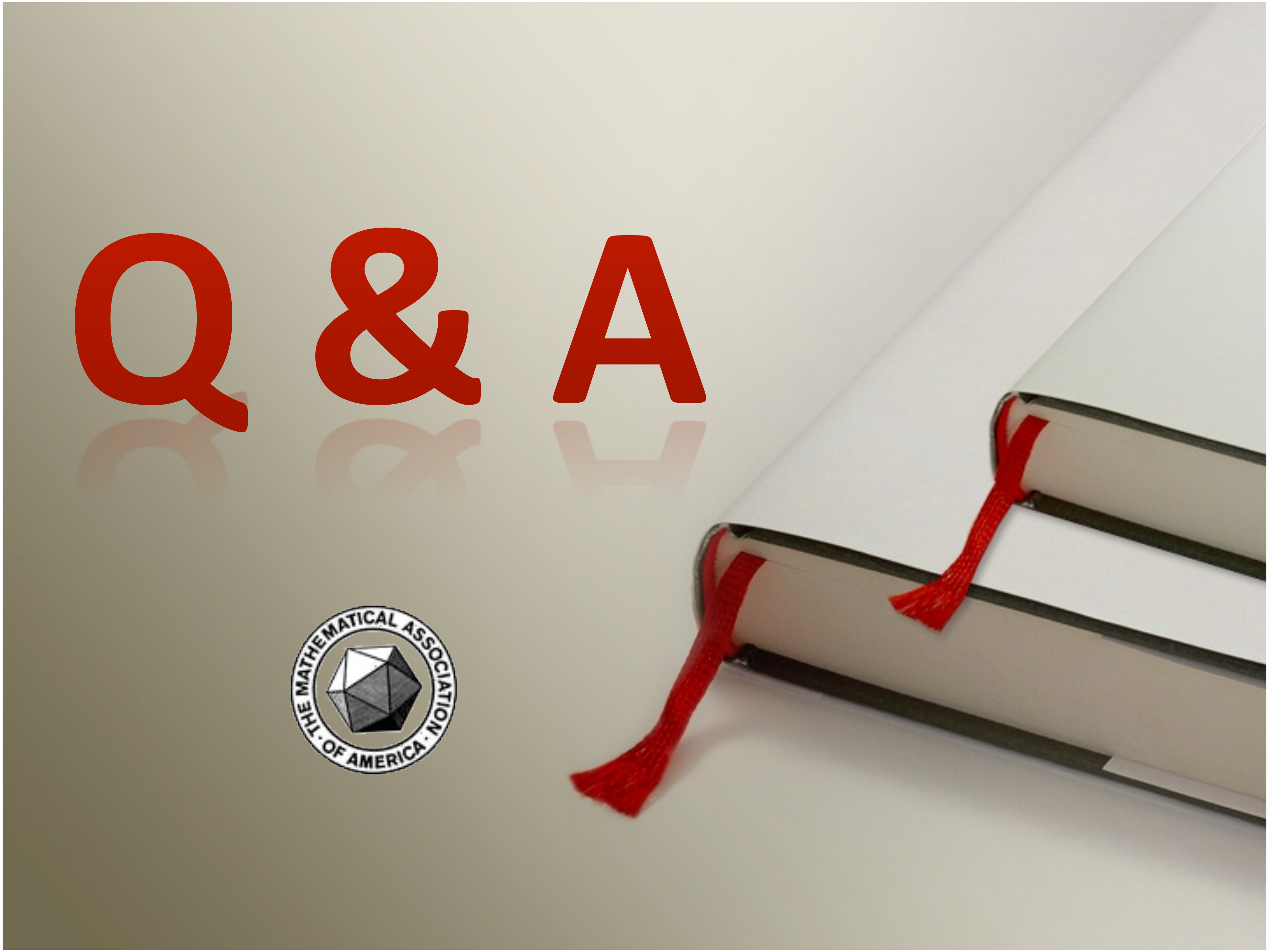
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